4章C問題（解答）

$\boxed{26}$ (1) $e^{-|x|} = \begin{cases} e^{-x} & (x \geqq 0) \\ e^{x} & (x < 0) \end{cases}$ なので，

$$F(\xi) = \int_{-\infty}^{\infty} e^{-|x|}e^{-i\xi x}dx = \int_{-\infty}^{0} e^{x}e^{-i\xi x}dx + \int_{0}^{\infty} e^{-x}e^{-i\xi x}dx$$

$$= \int_{-\infty}^{0} e^{(1-i\xi)x}dx + \int_{0}^{\infty} e^{(-1-i\xi)x}dx$$

$$= \left[\frac{1}{1-i\xi}e^{(1-i\xi)x}\right]_{-\infty}^{0} + \left[\frac{1}{-1-i\xi}e^{(-1-i\xi)x}\right]_{0}^{\infty}$$

ここで，$|e^{(1-i\xi)x}| = e^x$ であり，$\displaystyle\lim_{x\to-\infty} e^x = 0$ なので，$\displaystyle\lim_{x\to-\infty} e^{(1-i\xi)x} = 0$

したがって，$\displaystyle\left[\frac{1}{1-i\xi}e^{(1-i\xi)x}\right]_{-\infty}^{0} = \frac{1}{1-i\xi}$

同様にして $\displaystyle\left[\frac{1}{-1-i\xi}e^{(-1-i\xi)x}\right]_{0}^{\infty} = \frac{1}{1+i\xi}$ より $\displaystyle F(\xi) = \frac{2}{1+\xi^2}$

(2) $f(x)$ は連続なので，フーリエの積分定理より

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)e^{i\xi x}d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\xi^2}e^{i\xi x}d\xi$$

$\displaystyle f(1) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi}}{1+\xi^2}d\xi$，$\displaystyle f(-1) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\xi}}{1+\xi^2}d\xi$ であり

オイラーの公式から $e^{i\xi} + e^{-i\xi} = 2\cos\xi$ が成り立つから

$$f(1) + f(-1) = \frac{1}{\pi}\left(\int_{-\infty}^{\infty} \frac{e^{i\xi}}{1+\xi^2}d\xi + \int_{-\infty}^{\infty} \frac{e^{-i\xi}}{1+\xi^2}d\xi\right) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\cos\xi}{1+\xi^2}d\xi$$

となる．$\cos\xi$ は偶関数なので，

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\cos\xi}{1+\xi^2}d\xi = \frac{4}{\pi}\int_{0}^{\infty} \frac{\cos\xi}{1+\xi^2}d\xi = \frac{4}{\pi}\int_{0}^{\infty} \frac{\cos u}{1+u^2}du$$

一方，$f(x) = e^{-|x|}$ より $\displaystyle f(1) + f(-1) = \frac{2}{e}$

したがって，$\displaystyle\int_{0}^{\infty} \frac{\cos u}{1+u^2}du = \frac{\pi}{2e}$

$\boxed{27}$ 例題 4.2 と同様の計算により

$$\int_{-\pi}^{\pi} (f(x) - S_n(x))^2 \, dx = \int_{-\pi}^{\pi} f(x)^2 dx - \frac{\pi}{2} {a_0}^2 - \pi \sum_{k=1}^{n} ({a_k}^2 + {b_k}^2)$$

であり．$n \to \infty$ の時，フーリエの平均収束定理より左辺 $\to 0$ なので

$$\int_{-\pi}^{\pi} f(x)^2 dx - \frac{\pi}{2} {a_0}^2 - \pi \sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2) = 0$$

移項して両辺を π で割れば求める等式が得られる．

$\boxed{28}$ (1) $f(x)$ は偶関数なので，$b_n = 0 \ (n = 1, 2, 3, \ldots)$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x dx = \frac{2}{\pi} \int_0^{\pi} x(-\cos x)' dx$$

$$= \frac{2}{\pi} \left[-x \cos x \right]_0^{\pi} + \frac{2}{\pi} \int_0^{\pi} \cos x dx = 2 + \frac{2}{\pi} \left[\sin x \right]_0^{\pi} = 2$$

$$a_1 = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos x dx = \frac{1}{\pi} \int_0^{\pi} x (2 \sin x \cos x) dx = \frac{1}{\pi} \int_0^{\pi} x \sin 2x dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left(-\frac{1}{2} \cos 2x \right)' dx = \frac{1}{\pi} \left[-\frac{x}{2} \cos 2x \right]_0^{\pi} + \frac{1}{2\pi} \int_0^{\pi} \cos 2x dx$$

$$= -\frac{1}{2} + \frac{1}{2\pi} \left[\frac{1}{2} \sin 2x \right]_0^{\pi} = -\frac{1}{2}$$

$n \geqq 2$ のとき，

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx dx = \frac{1}{\pi} \int_0^{\pi} x \left\{ \sin(n+1)x - \sin(n-1)x \right\} dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left\{ -\frac{1}{n+1} \cos(n+1)x + \frac{1}{n-1} \cos(n-1)x \right\}' dx$$

$$= \frac{1}{\pi} \left[-\frac{x}{n+1} \cos(n+1)x + \frac{x}{n-1} \cos(n-1)x \right]_0^{\pi}$$

$$+ \frac{1}{\pi} \int_0^{\pi} \left\{ \frac{1}{n+1} \cos(n+1)x - \frac{1}{n-1} \cos(n-1)x \right\} dx$$

$$= -\frac{1}{n+1} (-1)^{n+1} + \frac{1}{n-1} (-1)^{n-1}$$

$$+ \frac{1}{\pi} \left[\frac{1}{(n+1)^2} \sin(n+1)x - \frac{1}{(n-1)^2} \sin(n-1)x \right]_0^{\pi}$$

$$= \frac{1}{n+1}(-1)^{n+2} - \frac{1}{n-1}(-1)^n = \frac{1}{n+1}(-1)^n - \frac{1}{n-1}(-1)^n$$
$$= -\frac{2(-1)^n}{n^2-1},$$

よって，$a_0 = 2,\ a_1 = -\dfrac{1}{2},\ a_n = -\dfrac{2(-1)^n}{n^2-1}\ (n \geqq 2)$,

$b_n = 0\ (n = 1, 2, 3, \dots)$

(2) パーセバルの等式より

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} {a_n}^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2 dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin^2 x dx$$
$$= \frac{1}{\pi}\int_0^{\pi} x^2(1-\cos 2x)dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx - \frac{1}{\pi}\int_0^{\pi} x^2 \cos 2x dx$$
$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi} x^2 \left(\frac{1}{2}\sin 2x\right)' dx$$
$$= \frac{\pi^2}{3} - \frac{1}{\pi}\left[\frac{1}{2}x^2 \sin 2x\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi} x \sin 2x dx$$
$$= \frac{\pi^2}{3} + \frac{1}{\pi}\int_0^{\pi} x\left(-\frac{1}{2}\cos 2x\right)' dx = \frac{\pi^2}{3} + \frac{1}{\pi}\left[-\frac{1}{2}x\cos 2x\right]_0^{\pi} + \frac{1}{2\pi}\int_0^{\pi} \cos 2x dx$$
$$= \frac{\pi^2}{3} - \frac{1}{2} + \frac{1}{2\pi}\left[\frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi^2}{3} - \frac{1}{2}$$

$\boxed{29}$ (1) $u_1(z) = \displaystyle\sum_{k=0}^{\infty} \hat{f}(k)z^k, u_2(\bar{z}) = \sum_{k=-1}^{-\infty} \hat{f}(k)\bar{z}^{-k}$ とおくと $u(z) = u_1(z) + u_2(\bar{z})$ である.

M を $[0,1]$ における $|f(t)|$ の最大値とすると

$$|\hat{f}(k)| = \left|\int_0^1 e^{-2\pi ikt} f(t)dt\right| \leqq \int_0^1 |e^{-2\pi ikt} f(t)|dt$$
$$\leqq \int_0^1 |e^{-2\pi ikt}||f(t)|dt = \int_0^1 |f(t)|dt \leqq M \text{ となる.}$$

無限等比級数の公式から,

$$|z|<1 \text{ のとき} \sum_{k=0}^{\infty} |f(k)||z|^k \leqq \sum_{k=0}^{\infty} M|z|^k = \frac{M}{1-|z|}$$

となるので $u_1(z)$ は絶対収束する．

$u_2(z)$ についても同様に絶対収束することがわかる．

(2) $\overline{u_1(z)} = \sum_{k=0}^{\infty} \overline{\hat{f}(k)}\bar{z}^k$ において $\hat{f}(0) = \int_0^1 f(t)dt$ より $\overline{\hat{f}(0)} = \hat{f}(0)$

および $\bar{\hat{f}}(k) = \int_0^1 \overline{e^{-2\pi ikt}f(t)}dt = \int_0^1 e^{2\pi ikt}f(t)dt = \hat{f}(-k)$ を用いると

$$\begin{aligned}\overline{u_1(z)} &= \sum_{k=1}^{\infty} \hat{f}(-k)\bar{z}^k = \hat{f}(0) + \sum_{k=1}^{\infty} \hat{f}(-k)\bar{z}^k = \hat{f}(0) + \sum_{k=-1}^{-\infty} \hat{f}(k)\bar{z}^{-k} \\ &= \hat{f}(0) + u_2(\bar{z}) \text{ となる．}\end{aligned}$$

両方の共役複素数をとると $u_1(z) = \hat{f}(0) + \overline{u_2(\bar{z})}$ となるので

$\overline{u_2(\bar{z})} = -\hat{f}(0) + u_1(z)$ である．よって

$\overline{u(z)} = \overline{u_1(z)} + \overline{u_2(\bar{z})} = \hat{f}(0) + u_2(\bar{z}) - \hat{f}(0) + u_1(z) = u_1(z) + u_2(\bar{z}) = u(z)$

したがって，$u(z)$ は実数値関数である．

$u_1(z), u_2(z)$ は $|z|<1$ において絶対収束するので，$u_1(z)$ は z で，$u_2(z)$ は $\bar{z}$ でそれぞれ項別微分可能である．x, y のかわりに $z, \bar{z}$ を独立変数として考えれば,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \sum_{k=0}^{\infty} \hat{f}(k)kz^{k-1}\frac{\partial z}{\partial x} + \sum_{k=-1}^{-\infty} \hat{f}(k)(-k)\bar{z}^{-k-1}\frac{\partial \bar{z}}{\partial x} \\ &= \sum_{k=0}^{\infty} \hat{f}(k)kz^{k-1} + \sum_{k=-1}^{-\infty} \hat{f}(k)(-k)\bar{z}^{-k-1},\end{aligned}$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{k=0}^{\infty} \hat{f}(k)k(k-1)z^{k-2} + \sum_{k=-1}^{-\infty} \hat{f}(k)(-k)(-k-1)z^{-k-2}$$

同様に

$$\frac{\partial u}{\partial y} = \sum_{k=0}^{\infty} \hat{f}(k)kz^{k-1}\frac{\partial z}{\partial y} + \sum_{k=-1}^{-\infty} \hat{f}(k)(-k)\bar{z}^{-k-1}\frac{\partial \bar{z}}{\partial y}$$

$$= i\sum_{k=0}^{\infty}\hat{f}(k)kz^{k-1} - i\sum_{k=-1}^{-\infty}\hat{f}(k)(-k)\bar{z}^{-k-1},$$

$$\frac{\partial^2 u}{\partial y^2} = -\sum_{k=0}^{\infty}\hat{f}(k)k(k-1)z^{k-2} - \sum_{k=-1}^{-\infty}\hat{f}(k)(-k)(-k-1)\bar{z}^{-k-2}$$

となるので，$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(3) $z = re^{2\pi i\theta}$，$\bar{z} = re^{-2\pi i\theta}$ より $z^k = r^k e^{2\pi ik\theta}$，$\bar{z}^{-k} = r^{-k}e^{2\pi ik\theta}$ なので，

$$u(z) = \sum_{k=0}^{\infty}\int_0^1 e^{-2\pi ikt}f(t)dt\ r^k e^{2\pi ik\theta} + \sum_{k=-1}^{-\infty}\int_0^1 e^{-2\pi ikt}f(t)dt\ r^{-k}e^{2\pi ik\theta}$$

$$= \int_0^1 f(t)\left(\sum_{k=0}^{\infty} r^k e^{2\pi ik(\theta-t)}\right)dt + \int_0^1 f(t)\left(\sum_{k=-1}^{-\infty} r^{-k}e^{2\pi ik(\theta-t)}\right)dt$$

ここで，無限等比級数の公式から $\displaystyle\sum_{k=0}^{\infty} r^k e^{2\pi ik(\theta-t)} = \frac{1}{1-re^{2\pi i(\theta-t)}}$，

$\displaystyle\sum_{k=-1}^{-\infty} r^{-k}e^{2\pi ik(\theta-t)} = \sum_{k=1}^{\infty} r^k e^{-2\pi ik(\theta-t)} = \frac{re^{-2\pi i(\theta-t)}}{1-re^{-2\pi i(\theta-t)}}$ が成り立つので，

$$u(z) = \int_0^1 f(t)\left(\frac{1}{1-re^{2\pi i(\theta-t)}} + \frac{re^{-2\pi i(\theta-t)}}{1-re^{-2\pi i(\theta-t)}}\right)dt$$

$$= \int_0^1 f(t)\frac{1-re^{-2\pi i(\theta-t)} + re^{-2\pi i(\theta-t)}(1-re^{2\pi i(\theta-t)})}{1-r\left(e^{2\pi i(\theta-t)}+e^{-2\pi i(\theta-t)}\right)+r^2}dt$$

$$= \int_0^1 f(t)\frac{1-r^2}{1-2r\cos(2\pi(\theta-t))+r^2}dt$$