73   以下，C は積分定数である．

(1) 置換積分法を用いる．

$t = \sqrt{2x+1}$ とおくと，$t^2 = 2x+1 \Leftrightarrow x = \dfrac{t^2-1}{2}$ であり，$dx = tdt$

$$\int \frac{3x}{\sqrt{2x+1}}\,dx = \frac{1}{2}\int \frac{3(t^2-1)}{t}\cdot tdt = \frac{3}{2}\int (t^2-1)dt$$

$$= \frac{3}{2}\left(\frac{t^3}{3}-t\right)+C$$

$$= \frac{t}{2}\left(t^2-3\right)+C$$

$$= \frac{\sqrt{2x+1}}{2}\{(2x+1)-3\}+C$$

$$= (x-1)\sqrt{2x+1}+C$$

(2) $\dfrac{1}{x(2-x)} = \dfrac{a}{x}+\dfrac{b}{x-2}$ とおくと，$-1 = a(x-2)+bx$

したがって $a = \dfrac{1}{2},\ \ b = -\dfrac{1}{2}$

$$\int \frac{1}{x(2-x)}\,dx = \frac{1}{2}\int \left(\frac{1}{x}-\frac{1}{x-2}\right)dx$$

$$= \frac{1}{2}\left(\log|x|-\log|x-2|\right)+C = \frac{1}{2}\log\left|\frac{x}{x-2}\right|+C$$

(3) 置換積分法を用いる．

$t = \log x$　とおくと，$dt = \dfrac{1}{x}dx$　であるから

$$(与式) = \int \frac{1}{t}dt = \log|t| + C = \log\left|\log x\right| + C$$

(4) 部分積分法を用いる．

$$\int \frac{\log x}{x^2}dx = \int \left(-\frac{1}{x}\right)' \log x \, dx = -\frac{\log x}{x} - \int \left(-\frac{1}{x}\right)(\log x)' \, dx$$

$$= -\frac{\log x}{x} + \int \frac{1}{x^2}dx$$

$$= -\frac{\log x}{x} - \frac{1}{x} + C$$

$$= -\frac{1}{x}(\log x + 1) + C$$

(5) $\dfrac{x^4}{x^3-1} = \dfrac{x(x^3-1)+x}{x^3-1} = x + \dfrac{x}{x^3-1} = x + \dfrac{x}{(x-1)(x^2+x+1)}$ であるから

$\dfrac{x}{(x-1)(x^2+x+1)} = \dfrac{a}{x-1} + \dfrac{bx+c}{x^2+x+1}$ とおくと，$a = c = \dfrac{1}{3}$，$b = -\dfrac{1}{3}$

$\dfrac{x-1}{x^2+x+1} = \dfrac{1}{2}\left(\dfrac{2x-2}{x^2+x+1}\right) = \dfrac{1}{2}\left(\dfrac{2x+1-3}{x^2+x+1}\right) = \dfrac{1}{2}\left(\dfrac{(x^2+x+1)'}{x^2+x+1} - 3\dfrac{1}{x^2+x+1}\right)$ より

$$\int \dfrac{x^4}{x^3-1}\,dx = \int\left\{x + \dfrac{1}{3}\left(\dfrac{1}{x-1} - \dfrac{x-1}{x^2+x+1}\right)\right\}dx$$

$$= \int\left\{x + \dfrac{1}{3}\left(\dfrac{1}{x-1} - \dfrac{1}{2}\left(\dfrac{(x^2+x+1)'}{x^2+x+1} - 3\dfrac{1}{x^2+x+1}\right)\right)\right\}dx$$

$$= \dfrac{1}{2}x^2 + \dfrac{1}{3}\left\{\log|x-1| - \dfrac{1}{2}\log(x^2+x+1) + \dfrac{3}{2}\left(\dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2}{\sqrt{3}}\left(x+\dfrac{1}{2}\right)\right)\right\} + C$$

$$= \dfrac{1}{2}x^2 + \dfrac{1}{3}\log\dfrac{|x-1|}{\sqrt{x^2+x+1}} + \dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2}{\sqrt{3}}\left(x+\dfrac{1}{2}\right) + C$$

補足 $\dfrac{1}{x^2+x+1} = \dfrac{1}{\left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4}}$ より，$t = x + \dfrac{1}{2}$ とおくと $dt = dx$

$$\int \dfrac{1}{x^2+x+1}\,dx = \int \dfrac{1}{\left(x+\dfrac{1}{2}\right)^2 + \dfrac{3}{4}}\,dx = \int \dfrac{1}{t^2+\dfrac{3}{4}}\,dt = \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2t}{\sqrt{3}} + C$$

$$= \dfrac{2}{\sqrt{3}}\tan^{-1}\dfrac{2}{\sqrt{3}}\left(x+\dfrac{1}{2}\right) + C$$

74 $x = \tan\theta$ とおくと,

$$1 + x^2 = 1 + \tan^2\theta = \frac{1}{\cos^2\theta}, \quad dx = \frac{1}{\cos^2\theta}d\theta,$$

x	$0 \to 1$
θ	$0 \to \dfrac{\pi}{4}$

より

$$I = \int_0^1 \frac{\log(x+1)}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{\log(\tan\theta+1)}{\dfrac{1}{\cos^2\theta}} \cdot \frac{1}{\cos^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \log(\tan\theta+1)\,d\theta$$

ここで, $\log(\tan\theta+1) = \log\dfrac{\sin\theta+\cos\theta}{\cos\theta} = \log\dfrac{\sqrt{2}\cos\left(\dfrac{\pi}{4}-\theta\right)}{\cos\theta}$

$$= \log\sqrt{2} + \log\cos\left(\frac{\pi}{4}-\theta\right) - \log\cos\theta \quad \text{を代入すると}$$

$$I = \int_0^{\frac{\pi}{4}} \log\sqrt{2}\,d\theta + \int_0^{\frac{\pi}{4}} \log\cos\left(\frac{\pi}{4}-\theta\right)d\theta - \int_0^{\frac{\pi}{4}} \log\cos\theta\,d\theta$$

さらに, $t = \dfrac{\pi}{4} - \theta$ とおくと

$$\int_0^{\frac{\pi}{4}} \log\cos\left(\frac{\pi}{4}-\theta\right)d\theta = \int_0^{\frac{\pi}{4}} \log\cos t\,dt = \int_0^{\frac{\pi}{4}} \log\cos\theta\,d\theta \quad \text{だから}$$

$$I = \log\sqrt{2}\int_0^{\frac{\pi}{4}} 1\,d\theta + \int_0^{\frac{\pi}{4}} \log\cos\theta\,d\theta - \int_0^{\frac{\pi}{4}} \log\cos\theta\,d\theta = \frac{\log 2}{2}\big[\theta\big]_0^{\frac{\pi}{4}} = \frac{\pi\log 2}{8} \quad \square$$

75 (1) $t = \tan^{-1}x$ とおくと, $dt = \dfrac{1}{1+x^2}dx$,

x	$0 \to 1$
t	$0 \to \dfrac{\pi}{4}$

より

$$\int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} t\,dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{\pi^2}{32}$$

4

(2) $I = \int_0^1 x\left(\tan^{-1} x\right)^2 \, dx$ とおくと, $I = \left[\dfrac{x^2}{2}\left(\tan^{-1} x\right)^2\right]_0^1 - \int_0^1 \dfrac{x^2}{1+x^2}\left(\tan^{-1} x\right) dx$

ここで, $\dfrac{x^2}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = 1 - \dfrac{1}{1+x^2}$ であるから

$I = \int_0^1 x\left(\tan^{-1} x\right)^2 \, dx = \dfrac{\pi^2}{32} - \int_0^1\left(1 - \dfrac{1}{1+x^2}\right)\left(\tan^{-1} x\right) dx = \dfrac{\pi^2}{32} - \int_0^1 \tan^{-1} x \, dx + \int_0^1 \dfrac{\tan^{-1} x}{1+x^2} \, dx$

(1) より $\int_0^1 \dfrac{\tan^{-1} x}{1+x^2} \, dx = \dfrac{\pi^2}{32}$, また,

$\int_0^1 \tan^{-1} x \, dx = \left[x\tan^{-1} x\right]_0^1 - \dfrac{1}{2}\int_0^1 \dfrac{2x}{1+x^2} \, dx = \dfrac{\pi}{4} - \dfrac{1}{2}\left[\log\left(x^2+1\right)\right]_0^1 = \dfrac{\pi}{4} - \dfrac{1}{2}\log 2$

であるから

$I = \dfrac{\pi^2}{32} - \left(\dfrac{\pi}{4} - \dfrac{1}{2}\log 2\right) + \dfrac{\pi^2}{32} = \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2$

別解 $I = \int_0^1 x\left(\tan^{-1} x\right)^2 \, dx = \int_0^1 \left(\dfrac{x^2+1}{2}\right)'\left(\tan^{-1} x\right)^2 \, dx$

$= \left[\dfrac{x^2+1}{2}\left(\tan^{-1} x\right)^2\right]_0^1 - \int_0^1 \dfrac{x^2+1}{2} \cdot 2\tan^{-1} x \cdot \dfrac{1}{x^2+1} \, dx$

部分積分法を
2 回適用

$= \left(\dfrac{\pi}{4}\right)^2 - \int_0^1 \tan^{-1} x \, dx$

$= \dfrac{\pi^2}{16} - \int_0^1 (x)' \tan^{-1} x \, dx$

$= \dfrac{\pi^2}{16} - \left(\left[x\tan^{-1} x\right]_0^1 - \int_0^1 \dfrac{x}{x^2+1} \, dx\right)$

$= \dfrac{\pi^2}{16} - \left(\dfrac{\pi}{4} - \int_0^1 \dfrac{x}{x^2+1} \, dx\right)$

$= \dfrac{\pi^2}{16} - \left(\dfrac{\pi}{4} - \left[\dfrac{1}{2}\log(x^2+1)\right]_0^1\right)$

$= \dfrac{\pi^2}{16} - \dfrac{\pi}{4} + \dfrac{1}{2}\log 2$

5

(3) $x = \sin\theta$ とおくと，$dx = \cos\theta\, d\theta$，

x	$0 \to \dfrac{1}{2}$
θ	$0 \to \dfrac{\pi}{6}$

より，

この範囲では $\cos\theta > 0$ に注意すると

$\sqrt{1 - x^2} = \sqrt{1 - \sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$ だから，

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1 - x^2}}\, dx = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta} \cos\theta\, d\theta = \int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 2\theta)\, d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} = \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

(4) $x = \sin\theta$ とおくと，$\sin^{-1} x = \sin^{-1}(\sin\theta) = \theta$，$dx = \cos\theta\, d\theta$，

x	$0 \to 1$
θ	$0 \to \dfrac{\pi}{2}$

より，

$$\int_0^1 x^2 \sin^{-1} x\, dx = \int_0^{\frac{\pi}{2}} \sin^2\theta \cos\theta \cdot \theta\, d\theta = \frac{1}{3}\left[\sin^3\theta \cdot \theta\right]_0^{\frac{\pi}{2}} - \frac{1}{3}\int_0^{\frac{\pi}{2}} \sin^3\theta\, d\theta = \frac{\pi}{6} - \frac{1}{3}\cdot\frac{2}{3}\cdot 1 = \frac{\pi}{6} - \frac{2}{9}$$
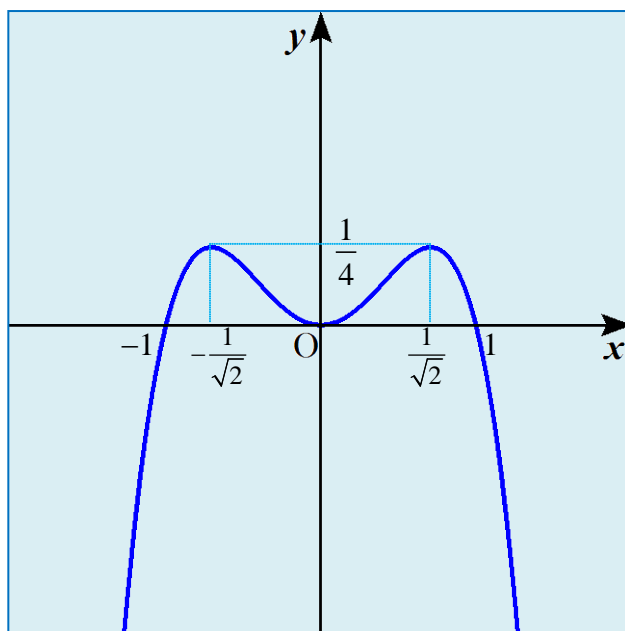
$\boxed{76}$ (1) $f(x) = x^2(1-x^2) = x^2 - x^4$ より

$$f'(x) = -4x^3 + 2x = -4x\left(x^2 - \frac{1}{2}\right) = -4x\left(x + \frac{1}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}}\right)$$
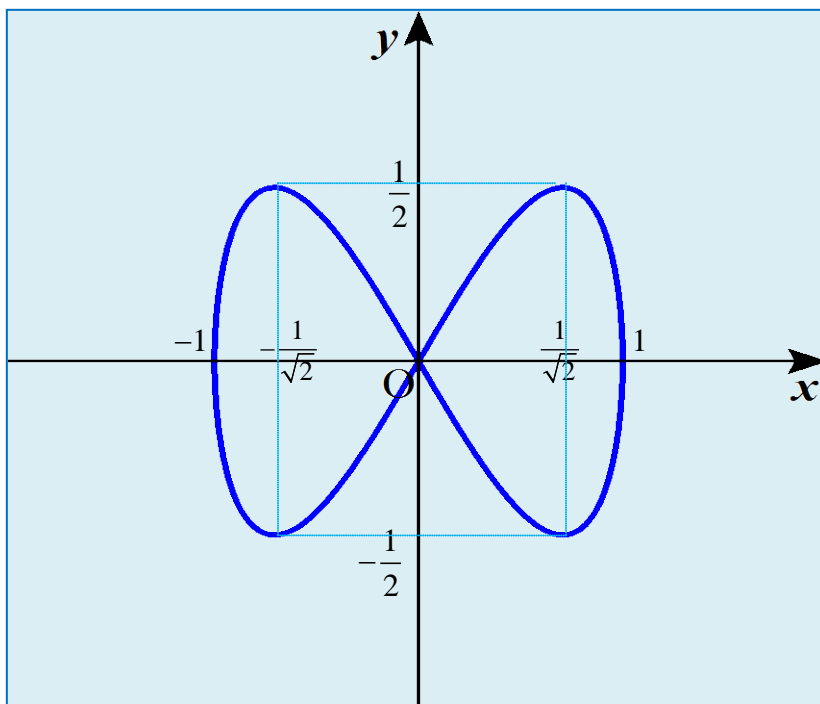
よって $f(x)$ の増減表は次のようになる.

x	$\cdots$	$-\dfrac{1}{\sqrt{2}}$	$\cdots$	0	$\cdots$	$\dfrac{1}{\sqrt{2}}$	$\cdots$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$	$\nearrow$	$\dfrac{1}{4}$	$\searrow$	0	$\nearrow$	$\dfrac{1}{4}$	$\searrow$

$f(x)$ のグラフは y 軸対称であり，グラフは下図のようになる.

(2) $y^2 = x^2(1-x^2) \geqq 0$ より $-1 \leqq x \leqq 1$ の部分のみ，実数 y の値が存在する．

(3) $V_x = 2\int_0^1 \pi y^2 \ dx = 2\pi \int_0^1 (x^2 - x^4) \ dx = 2\pi \left[\dfrac{x^3}{3} - \dfrac{x^5}{5} \right]_0^1 = \dfrac{4}{15}\pi$

$y^2 = x^2 - x^4$ より， $(x^2)^2 - x^2 + y^2 = 0$ であるから $x^2 = \dfrac{1 \pm \sqrt{1-4y^2}}{2}$

D は x 軸対称だから

$V_y = 2\pi \displaystyle\int_0^{\frac{1}{2}} \dfrac{1+\sqrt{1-4y^2}}{2} \ dy - 2\pi \int_0^{\frac{1}{2}} \dfrac{1-\sqrt{1-4y^2}}{2} \ dy$

$\qquad = 2\pi \displaystyle\int_0^{\frac{1}{2}} \sqrt{1-4y^2} \ dy = 4\pi \int_0^{\frac{1}{2}} \sqrt{\dfrac{1}{4} - y^2} \ dy$

$\qquad = 4\pi \cdot \dfrac{\pi}{4} \cdot \left(\dfrac{1}{2} \right)^2 = \dfrac{\pi^2}{4}$

注意：円 $x^2 + y^2 = \dfrac{1}{4}$

の面積の $\dfrac{1}{4}$ に等しい

$\boxed{77}$　（ア）　$b \geqq a > 0$ のとき

$$x^2 + (y-b)^2 = a^2 \Leftrightarrow (y-b)^2 = a^2 - x^2 \Leftrightarrow y = b \pm \sqrt{a^2 - x^2} \cdots ① \quad （複号同順）$$

$$V = \pi \left(\int_{-a}^{a} (b + \sqrt{a^2 - x^2})^2 dx - \int_{-a}^{a} (b - \sqrt{a^2 - x^2})^2 dx \right)$$

$$= 4\pi b \int_{-a}^{a} \sqrt{a^2 - x^2} \, dx$$

$$= 4\pi b \cdot \frac{1}{2} \cdot \pi a^2$$

$$= 2\pi^2 a^2 b$$

次に，①の両辺を x で微分すると　$y' = \mp \dfrac{x}{\sqrt{a^2 - x^2}}$　（複号同順）

$$\sqrt{1 + (y')^2} = \sqrt{1 + \frac{x^2}{a^2 - x^2}} = \frac{a}{\sqrt{a^2 - x^2}}　より，表面積は$$

$$S = \int_{-a}^{a} 2\pi (b + \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}} dx + \int_{-a}^{a} 2\pi (b - \sqrt{a^2 - x^2}) \frac{a}{\sqrt{a^2 - x^2}} dx$$

$$= 8\pi ab \int_{0}^{a} \frac{dx}{\sqrt{a^2 - x^2}} = 8\pi ab \left[\sin^{-1} \frac{x}{a} \right]_{0}^{a}$$

$$= 8\pi ab \cdot \frac{\pi}{2} = 4\pi^2 ab$$

（イ）　$a > b > 0$ のとき

$$V = 2\pi \left(\int_{0}^{a} (b + \sqrt{a^2 - x^2})^2 dx - \int_{\sqrt{a^2 - b^2}}^{a} (b - \sqrt{a^2 - x^2})^2 dx \right)$$

$$= 2\pi \left(\int_{0}^{\sqrt{a^2 - b^2}} (b^2 + a^2 - x^2) dx + \int_{0}^{a} 2b\sqrt{a^2 - x^2} \, dx + \int_{\sqrt{a^2 - b^2}}^{a} 2b\sqrt{a^2 - x^2} \, dx \right)$$

$$= 2\pi \left[(a^2 + b^2)x - \frac{x^3}{3} \right]_{0}^{\sqrt{a^2 - b^2}}$$

$$+ 4\pi b \cdot \frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right]_{0}^{a} + 4\pi b \cdot \frac{1}{2} \left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right]_{\sqrt{a^2 - b^2}}^{a}$$

$$= 2\pi \left\{ \frac{2a^2+b^2}{3}\sqrt{a^2-b^2} + a^2 b \left(\pi - \sin^{-1}\frac{\sqrt{a^2-b^2}}{a} \right) \right\}$$

$$S = 2\int_0^a 2\pi(b+\sqrt{a^2-x^2})\frac{a}{\sqrt{a^2-x^2}}dx + 2\int_{\sqrt{a^2-b^2}}^a 2\pi(b-\sqrt{a^2-x^2})\frac{a}{\sqrt{a^2-x^2}}dx$$

$$= 4\pi ab \left(\int_0^a \frac{dx}{\sqrt{a^2-x^2}} + \int_{\sqrt{a^2-b^2}}^a \frac{dx}{\sqrt{a^2-x^2}} \right) + 4\pi a \int_0^{\sqrt{a^2-b^2}} dx$$

$$= 4\pi ab \left(\left[\sin^{-1}\frac{x}{a} \right]_0^a + \left[\sin^{-1}\frac{x}{a} \right]_{\sqrt{a^2-b^2}}^a \right) + 4\pi a\sqrt{a^2-b^2}$$

$$= 4\pi a \left\{ \sqrt{a^2-b^2} + b \left(\pi - \sin^{-1}\frac{\sqrt{a^2-b^2}}{a} \right) \right\}$$

以上より,

■ $b \geqq a > 0$ のとき $V = 2\pi^2 a^2 b$, $S = 4\pi^2 ab$

■ $a > b > 0$ のとき

$$V = 2\pi \left\{ \frac{2a^2+b^2}{3}\sqrt{a^2-b^2} + a^2 b \left(\pi - \sin^{-1}\frac{\sqrt{a^2-b^2}}{a} \right) \right\}$$

$$S = 4\pi a \left\{ \sqrt{a^2-b^2} + b \left(\pi - \sin^{-1}\frac{\sqrt{a^2-b^2}}{a} \right) \right\}$$

$\boxed{78}$ (1) $S = \int_0^1 \{x - (-x)\}dx + \int_1^2 \{2 - x^2 - (-x)\}dx$

$$= \left[x^2\right]_0^1 + \left[-\frac{x^3}{3} + \frac{x^2}{2} + 2x\right]_1^2 = \frac{13}{6}$$

(2) $V_x = \pi \int_0^2 x^2 dx - \pi \int_{\sqrt{2}}^2 (2 - x^2)^2 dx$

$$= \pi \left[\frac{x^3}{3}\right]_0^2 - \pi \left[\frac{x^5}{5} - \frac{4x^3}{3} + 4x\right]_{\sqrt{2}}^2 = \frac{16(2\sqrt{2} - 1)}{15}\pi$$

(3) $V_y = \pi \int_{-2}^1 (2 - y)dy - \frac{1}{3}\pi \cdot 1^2 \cdot 1 - \frac{1}{3}\pi \cdot 2^2 \cdot 2$

$$= \pi \left[2y - \frac{y^2}{2}\right]_{-2}^1 - 3\pi = \frac{9}{2}\pi$$

(4) 曲線 $y = 2 - x^2$ $(1 \leqq x \leqq 2)$ 上の任意の点を，

P$(t, 2-t^2)$ $(1 \leqq t \leqq 2)$ とおく．P から直線 $y = -x$ に下ろした垂線の足を Q とおく．Q の座標は，直線 PQ と 直線 $y = -x$ の交点であるから，

連立方程式 $\begin{cases} y - (2 - t^2) = x - t \\ y = -x \end{cases}$ の解である．

これを解いて $Q\left(\dfrac{t^2 + t - 2}{2}, \ -\dfrac{t^2 + t - 2}{2} \right)$ を得る．

t が t から $t + dt$（dt 微小量）まで変化するとき，Q の動く距離を ds とすると

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{\left(\frac{2t+1}{2}\right)^2 + \left(-\frac{2t+1}{2}\right)^2}\, dt$$

$$= \sqrt{2}\left|\frac{2t+1}{2}\right| dt = \frac{2t+1}{\sqrt{2}}\, dt \quad (\text{注意}: \ 1 \leqq t \leqq 2 \ \text{より} \ 2t+1 > 0)$$

$$PQ = \sqrt{\left(t - \frac{t^2 + t - 2}{2}\right)^2 + \left(2 - t^2 + \frac{t^2 + t - 2}{2}\right)^2} = \frac{1}{\sqrt{2}}\sqrt{(t^2 - t - 2)^2}$$

したがって

$$V = \int_1^2 \pi \cdot PQ^2 \frac{ds}{dt}\, dt = \int_1^2 \frac{\pi}{2}(t^2 - t - 2)^2 \frac{(2t+1)}{\sqrt{2}}\, dt$$

$$= \frac{\pi}{2\sqrt{2}} \int_1^2 (t+1)^2 (t-2)^2 (2t+1)\, dt$$

ここで，$u = t - 2$ とおくと，$t = u + 2$，$dt = du$

$$V = \frac{\pi}{2\sqrt{2}} \int_{-1}^0 u^2 (u+3)^2 (2u+5)\, du$$

$$= \frac{\pi}{2\sqrt{2}} \int_{-1}^0 (2u^5 + 17u^4 + 48u^3 + 45u^2)\, du$$

$$= \frac{\pi}{2\sqrt{2}} \left[\frac{1}{3}u^6 + \frac{17}{5}u^5 + 12u^4 + 15u^3 \right]_{-1}^0$$

$$= \frac{91}{60}\sqrt{2}\pi$$

$\boxed{79}$ ■ $0 \leqq h \leqq 1$ のとき $V(h) = \pi \int_0^h x^2 \, dy = \pi \int_0^h y^2 \, dy = \pi \left[\dfrac{y^3}{3} \right]_0^h = \dfrac{\pi}{3} h^3$

特に $h = 1$ のとき, $V = \dfrac{\pi}{3}$

■ $h \geqq 1$ のとき

$V(h) = \dfrac{\pi}{3} + \pi \int_1^h \left(\dfrac{1}{y} \right)^2 dy = \dfrac{\pi}{3} + \pi \left[-\dfrac{1}{y} \right]_1^h = \pi \left(\dfrac{4}{3} - \dfrac{1}{h} \right)$ （注意： $V < \dfrac{4}{3}\pi$ ）

さて，1秒あたり単位体積の水が入っているから，t 秒後の高さ $h(t)$ は次のようになる．

（ア） $0 \leqq t \leqq \dfrac{\pi}{3}$ のとき $\dfrac{\pi}{3} h(t)^3 = t$ より $h(t) = \left(\dfrac{3t}{\pi} \right)^{\frac{1}{3}}$

（イ） $\dfrac{\pi}{3} \leqq t < \dfrac{4}{3}\pi$ のとき $\pi \left(\dfrac{4}{3} - \dfrac{1}{h(t)} \right) = t$ より $h(t) = \dfrac{3\pi}{4\pi - 3t}$

(2) (1)より $0 < t < \dfrac{\pi}{3}$ のとき $\dfrac{dh}{dt} = \dfrac{1}{3} \left(\dfrac{3}{\pi} \right)^{\frac{1}{3}} t^{-\frac{2}{3}}$

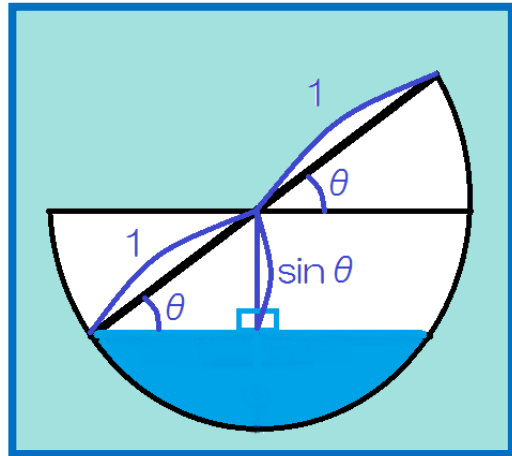$\dfrac{\pi}{3} < t < \dfrac{4}{3}\pi$ のとき $\dfrac{dh}{dt} = \dfrac{9\pi}{(4\pi - 3t)^2}$

80 (1) $V = \pi \int_{1-h}^{1} y^2 dx = \pi \int_{1-h}^{1} (1-x^2) dx = \pi \left[x - \dfrac{x^3}{3} \right]_{1-h}^{1} = \pi \left(h^2 - \dfrac{h^3}{3} \right)$
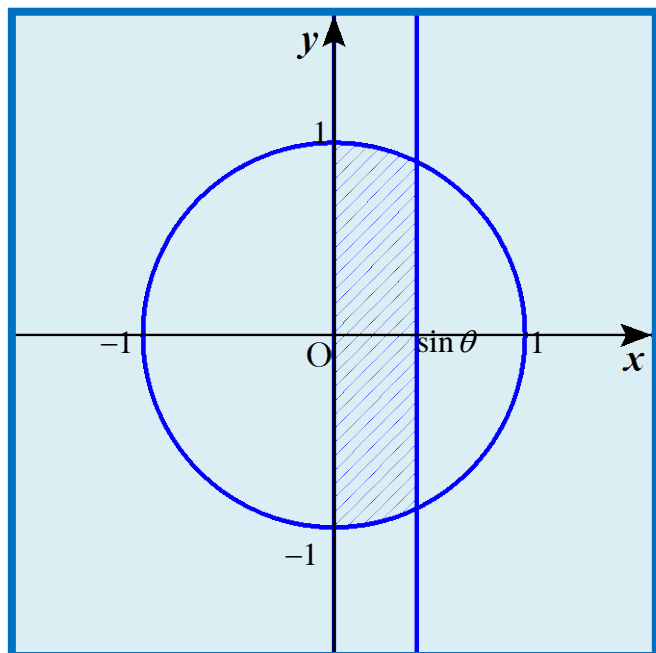
(2) こぼれる水の量は，図より

$V = \pi \displaystyle\int_{0}^{\sin\theta} y^2 dx = \pi \int_{0}^{\sin\theta} (1-x^2) dx$

$\quad = \pi \left[x - \dfrac{x^3}{3} \right]_{0}^{\sin\theta} = \pi \left(\sin\theta - \dfrac{\sin^3\theta}{3} \right)$

$x^2 + y^2 = 1 \Leftrightarrow y^2 = 1 - x^2$

81 (1) 頂点 $(0, b)$ より，放物線は $y = kx^2 + b$ …① とおける．

$x = \pm a$ のとき，$y = 0$ であるから，①へ代入して $0 = ka^2 + b$ $\therefore k = -\dfrac{b}{a^2}$

したがって，$y = -\dfrac{b}{a^2}x^2 + b$ …② を得る．
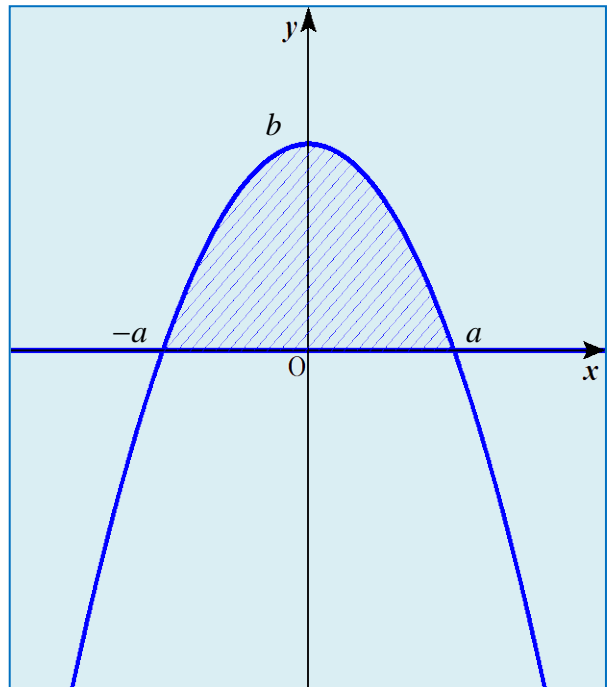
②より，$\dfrac{dy}{dx} = -\dfrac{2b}{a^2}x$

C は y 軸対称であるから，

$$L = 2\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\int_0^a \sqrt{1 + \left(-\frac{2b}{a^2}x\right)^2}\,dx = 2\int_0^a \frac{2b}{a^2}\sqrt{x^2 + \frac{a^4}{4b^2}}\,dx$$

$$= \frac{4b}{a^2}\frac{1}{2}\left[x\sqrt{x^2 + \frac{a^4}{4b^2}} + \frac{a^4}{4b^2}\log\left|x + \sqrt{x^2 + \frac{a^4}{4b^2}}\right|\right]_0^a$$

$$= \sqrt{a^2 + 4b^2} + \frac{a^2}{2b}\log\left(\frac{2b + \sqrt{a^2 + 4b^2}}{a}\right)$$

(2) ②より $x^2 = -\dfrac{a^2}{b}(y-b)$

この両辺を y で微分すると $2x\dfrac{dx}{dy} = -\dfrac{a^2}{b}$

したがって，

$$S = 2\pi \int_0^b |x|\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}\,dy = 2\pi \int_0^b |x|\sqrt{1+\left(-\dfrac{a^2}{2bx}\right)^2}\,dy$$

$$= 2\pi \int_0^b \sqrt{x^2 + \dfrac{a^4}{4b^2}}\,dy = 2\pi \int_0^b \sqrt{-\dfrac{a^2}{b}(y-b) + \dfrac{a^4}{4b^2}}\,dy$$

$$= 2\pi \int_0^b \sqrt{\dfrac{a^2\{(-4by+4b^2)+a^2\}}{4b^2}}\,dy = 2\pi \dfrac{a}{2b}\int_0^b \sqrt{-4by+4b^2+a^2}\,dy$$

$$= \dfrac{a}{b}\pi \left[\dfrac{2}{3}\dfrac{1}{-4b}\left(-4by+4b^2+a^2\right)^{\frac{3}{2}}\right]_0^b = -\dfrac{\pi a}{6b^2}\left[\left(-4by+4b^2+a^2\right)^{\frac{3}{2}}\right]_0^b$$

$$= \dfrac{\pi a}{6b^2}\left\{\left(a^2+4b^2\right)^{\frac{3}{2}} - a^3\right\}$$

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82　$2x^2 - 2xy + y^2 = 4$ …①　$\Leftrightarrow y^2 - 2xy + (2x^2 - 4) = 0 \Leftrightarrow y = x \pm \sqrt{4 - x^2}$

　　ここで，$4 - x^2 \geqq 0$　より　$-2 \leqq x \leqq 2$
　　したがって

$$\boxed{\text{注意：円 } x^2 + y^2 = 4 \text{ の面積の半分}}$$

$$S = \int_{-2}^{2} \left\{ \left(x + \sqrt{4-x^2}\right) - \left(x - \sqrt{4-x^2}\right) \right\} dx = 2\int_{-2}^{2} \sqrt{4-x^2} \, dx$$

$$= 2 \cdot \frac{1}{2}\pi \cdot 2^2 = 4\pi$$

図形①を x 軸まわりに $180°$ 回転させた図形の方程式は，
(x, y) を $(x, -y)$ に変えると，$2x^2 + 2xy + y^2 = 4$ …② である．

①，②の連立方程式を解くと，①と②の共有点は　$(x, y) = (0, \pm 2),\ (\pm\sqrt{2}, 0)$

この回転体は，y 軸対称な立体であるから

$$V = 2\pi \int_{0}^{2} \left(x + \sqrt{4-x^2}\right)^2 dx - 2\pi \int_{\sqrt{2}}^{2} \left(x - \sqrt{4-x^2}\right)^2 dx$$

$$= 2\pi \int_{0}^{2} \left(4 + 2x\sqrt{4-x^2}\right) dx - 2\pi \int_{\sqrt{2}}^{2} \left(4 - 2x\sqrt{4-x^2}\right) dx$$

ここで，$t = \sqrt{4-x^2}$ とおくと　$t^2 = 4 - x^2,\ 2tdt = -2xdx \Leftrightarrow xdx = -tdt$

x	$0 \to 2$
t	$2 \to 0$

x	$\sqrt{2} \to 2$
t	$\sqrt{2} \to 0$

$V = 8\pi \left(\left[x\right]_0^2 - \left[x\right]_{\sqrt{2}}^2 \right) + 4\pi \int_2^0 (-t^2) dt + 4\pi \int_{\sqrt{2}}^0 (-t^2) dt$

$$V = 8\sqrt{2}\pi + 4\pi \left[\frac{t^3}{3}\right]_0^2 + 4\pi \left[\frac{t^3}{3}\right]_0^{\sqrt{2}}$$

$$= \frac{32(\sqrt{2}+1)}{3}\pi$$