工科のための 偏 微 分 方 程 式　演 習 問 題 略 解

第 1 章　**1.1**　$\sin(x+ct)$.

1.2　(1)　$f(x-ct)+\displaystyle\int_0^t h(x-c(t-s),s)\,ds$.

(2)　$f(x-ct)+\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} h\left(\frac{1}{2}(x-ct+s),\,\frac{1}{2c}(s-x+ct)\right)ds$，$\tau=(s-x+ct)/(2c)$ と置換積分して (1) の形に変形できる．

1.3　(1)　$e^y f(x-y)$　(2)　$g(y-x)e^x$ と計算されるが，$e^y\left\{g(-(x-y))e^{x-y}\right\}$ と (1) の形に変形できる．

1.4　$2x+y+\sin(2x+y)-\sin(x+y)$

1.5　$e^{y-2x}+x\log\{1+(y-2x)^2\}$

1.6　$e^{-y}f(x-2y)-\dfrac{e^{-y}}{10}\{\sin(x-2y)-3\cos(x-2y)\}+\dfrac{1}{10}\{\sin(x+y)-3\cos(x+y)\}$.

1.7　$e^y(x+2y-xy-y^2)$.

1.8　$e^{2x}(y-x)^2-e^x(y^2+2y+2)$.

1.9　$x^2e^{-2y}+1$

1.10　$xg(y/x)$.

1.11　$\sqrt{x^2+y^2}$.

1.12　$g\left(e^{-x}(y-x^2-2x-2)+2\right)$.

1.14　$-\log\left(xy^2-y^3+e^{-f(x-3y)}\right)$.

1.15　$\dfrac{e^{y^2/2}f(x-y)}{1+\{2+(y^2-2)e^{y^2/2}\}f(x-y)}$.

1.16　$\dfrac{y-x}{2e^x-1}$.

1.17　$\exp\left\{\dfrac{2y-x^2-2x}{2(x+1)}\right\}$.

1.18　$u(x,t)=\left(\dfrac{2x}{1+\sqrt{1+4tx}}\right)^2$.

1.20　$f'(x)\geq 0$ または $|f'(x)|<c$ であれば良い．$u(x,t)=\dfrac{ce^{-ct}x}{1+c-e^{-ct}}$.

第 2 章　**2.3**　(1)　$u_x=U_\xi+U_\tau$，$u_t=c\,(3U_\xi-2U_\tau)$.

$u_{xx}=U_{\xi\xi}+2U_{\xi\tau}+U_{\tau\tau}$，$u_{tt}=c^2\{9U_{\xi\xi}-12U_{\xi\tau}+4U_{\tau\tau}\}$，$u_{xt}=c\{3U_{\xi\xi}+U_{\xi\tau}-2U_{\tau\tau}\}$.

(2)　$0=u_{tt}(x,t)-cu_{xt}-6c^2u_{xx}(x,t)=-25c^2U_{\xi\tau}$，すなわち，$U_{\xi\tau}=0$.

(3)　$u(x,t)=U(\xi,\tau)=\varphi(\xi)+\psi(\tau)=\varphi(x+3ct)+\psi(x-2ct)$.

(4)　$u(x,t)=\dfrac{1}{5}\{2f\,(x+3ct)+3f(x-2ct)\}+\dfrac{1}{5c}\displaystyle\int_{x-2ct}^{x+3ct}g(s)\,ds$.

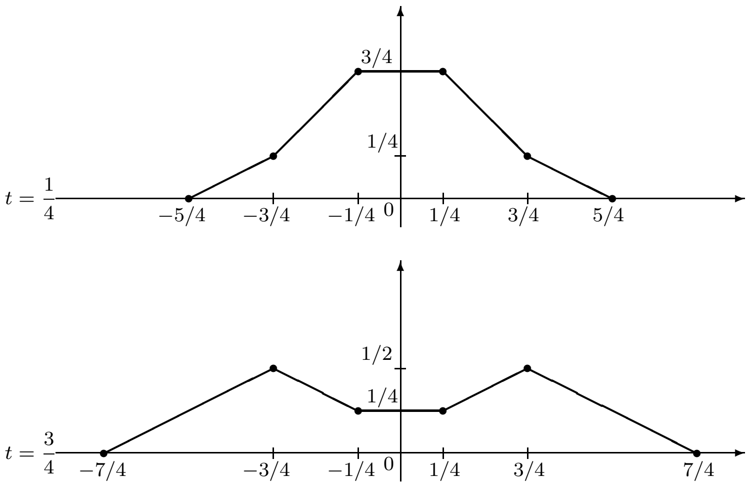
2.5　(1)　$c^2u_{xx}+cu_{xt}-2u_{tt}=\left(c\frac{\partial}{\partial x}+2\frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)u(x,t)$ だから $x=c\xi+c\tau$，$t=2\xi-\tau$ と

変数変換して，$u(x,t) = \dfrac{1}{3}\left\{f(x+ct) + 2f\left(x - \dfrac{c}{2}t\right)\right\} + \dfrac{2}{3c}\displaystyle\int_{x-(c/2)t}^{x+ct} g(s)\,ds.$

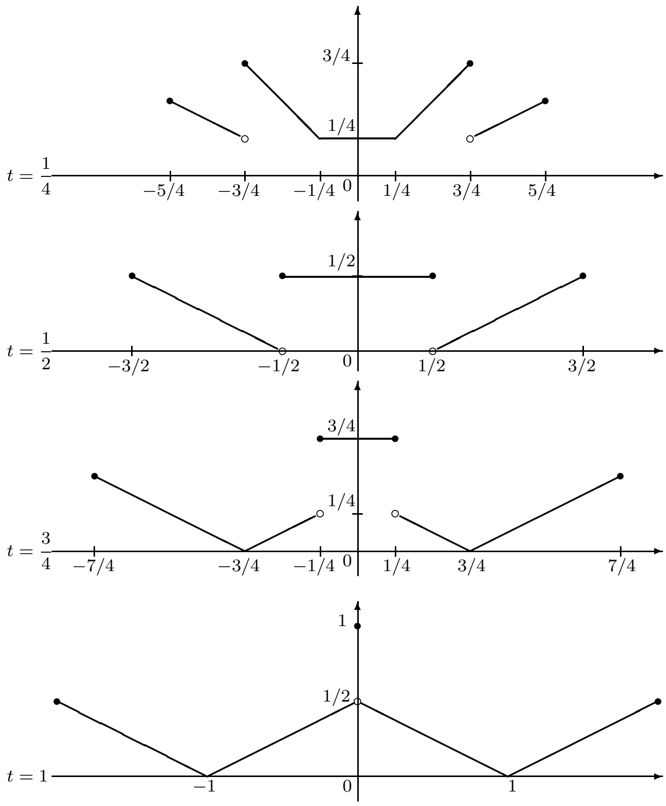
(2)　$c^2u_{xx} - 5cu_{xt} + 6u_{tt} = \left(c\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t}\right)\left(c\frac{\partial}{\partial x} - 3\frac{\partial}{\partial t}\right)u(x,t)$ だから，$x = c\xi + c\tau$，$t = -2\xi - 3\tau$ と変数変換して $u(x,t) = 3f\left(x + \dfrac{ct}{3}\right) - 2f\left(x + \dfrac{ct}{2}\right) + \dfrac{6}{c}\displaystyle\int_{x+(c/3)t}^{x+(c/2)t} g(s)\,ds.$

(3)　$c^2u_{xx} - 2cu_{xt} + u_{tt} = \left(c\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)^2 u(x,t)$ だから変数変換の仕方は一意的には定まらないが，例えば $x = c\xi + c\tau$，$t = -\xi + \tau$ と変数変換して，$u(x,t) = f(x+ct) - t\,(cf'(x+ct) - g(x+ct)).$

2.7



2.8



2.13 $\varphi(t,t)+\int_0^t \varphi_t(s,t)\,ds.$

2.15 (1) $\dfrac{1}{2(c^2-1)}\{\cos(x+t)+\cos(x-t)-\cos(x+ct)-\cos(x-ct)\}$（$c\neq 1$ のとき），
$\dfrac{t}{4}\{\sin(x+t)-\sin(x-t)\}$（$c=1$ のとき）.

(2) $\dfrac{1}{2c(c^2-1)}\{c(\sin(x+t)-\sin(x-t)))-(\sin(x+ct)-\sin(x-ct))\}$（$c\neq 1$ のとき），
$-\dfrac{t}{4}\{\cos(x+t)+\cos(x-t)\}+\dfrac{1}{4}\{\sin(x+t)-\sin(x-t)\}$（$c=1$ のとき）.

(3) $\dfrac{1}{2c(c^2+1)}\{2ce^{-t}\sin x+(\cos(x-ct)-c\sin(x-ct))-(\cos(x+ct)+c\sin(x+ct))\}.$

2.17 ディリクレ境界条件の場合には命題 2.3 の後の注意と同様にして計算でき，$h(x,t)/(2c)$ の $\Gamma_-^b(x_0,t_0)$ 上の二重積分で与えられる．ノイマン境界条件の場合には，さらに $\{x\geq 0,\ t\geq 0,\ x+ct\leq ct_0-x_0\}$ 上での $h(x,t)/c$ の二重積分を加えなければならない．第 6 章演習問題 6.20 を参照.

第 3 章 **3.1** (1) $\dfrac{1}{\sqrt{1+2ckt}}\exp\left\{-\dfrac{cx^2}{2(1+2ckt)}\right\}.$

(2) $x.$

(3) $x^2+2kt.$

(4) $x^3+6kxt.$

(5) $x^4+12ktx^2+12k^2t^2.$

3.2 $e^{-bt}\int_{-\infty}^{\infty}K(x-y,t)f(y)\,dy$

3.3 (1) $u(x,t)=e^{(2ax-a^2t)/(4k)}\int_{-\infty}^{\infty}K(x-y,t)e^{-ay/(2k)}f(y)\,dy$，ただし $\alpha=\frac{a}{2k}$，$\beta=-\frac{a^2}{4k}$.

(2) $u_t-ku_{xx}+au_x=u_s-ku_{yy}$，$u(x,t)=\int_{-\infty}^{\infty}K(x-at-y,t)f(y)\,dy.$

3.5 $\int_0^{\infty}\{K(x-y,t)-K(x+y,t)\}f(y)\,dy.$

3.6 $\mathrm{erf}(x)$ で誤差関数を表すとする：$\mathrm{erf}(x)=\dfrac{2}{\sqrt{\pi}}\int_0^x \exp(-y^2)\,dy.$

(1) $4xktK(x,t)+(x^2+2kt)\,\mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right).$

(2) $4ktK(x,t)(x^3+10ktx)+(12k^2t^2+12ktx^2+x^4)\,\mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right).$

(3) $e^{kt}\sinh x+\dfrac{e^{x+kt}}{2}\,\mathrm{erf}\left(\dfrac{x+2kt}{\sqrt{4kt}}\right)+\dfrac{e^{-x+kt}}{2}\,\mathrm{erf}\left(\dfrac{x-2kt}{\sqrt{4kt}}\right)-\mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right).$

3.7 $\int_0^{\infty}\{K(x-y,t)+K(x+y,t)\}f(y)\,dy.$

3.8 誤算関数 $\mathrm{erf}(x)$ に対して，

(1) $4ktK(x,t)+x\,\mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right).$

(2) $4kt(x^2+4kt)K(x,t)+(x^3+6kt)\,\mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right).$

(3) $e^{kt}\cosh x + \dfrac{e^{x+kt}}{2}\operatorname{erf}\left(\dfrac{x+2kt}{\sqrt{4kt}}\right) - \dfrac{e^{-x+kt}}{2}\operatorname{erf}\left(\dfrac{x-2kt}{\sqrt{4kt}}\right)$.

第 4 章 **4.5** $\lambda_n = \left\{\dfrac{(2n+1)\pi}{L}\right\}^2$, $u_n(x) = A_n \cos\dfrac{(2n+1)\pi x}{L} + B_n \sin\dfrac{(2n+1)\pi x}{L}$ $(n = 0, 1, 2, \dots)$.

4.7 $\lambda_n = \left\{\dfrac{(n+\frac{1}{2})\pi}{\ell}\right\}^2$, $X_n(x) = \cos\dfrac{(n+\frac{1}{2})\pi x}{\ell}$ $(n = 0, 1, 2, \dots)$.

4.8 $\lambda_{mn} = \left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)\pi^2$, $u_{mn}(x, y) = \cos\dfrac{m\pi x}{a}\cos\dfrac{n\pi x}{b}$ $(m = 0, 1, 2, \dots,\ n = 0, 1, 2, \dots)$.

4.9 $\lambda_{mn} = \left(\dfrac{m^2}{a^2} + \dfrac{n^2}{b^2}\right)\pi^2$, $u_{mn}(x, y) = \sin\dfrac{m\pi x}{a}\cos\dfrac{n\pi x}{b}$ $(m = 1, 2, \dots,\ n = 0, 1, 2, \dots)$.

4.10 n 次ベッセル関数 $J_n(x)$ の導関数 $J_n'(x)$ のゼロ点を小さいものから順番に $\tilde{j}_{n,1} < \tilde{j}_{n,2} < \cdots < \tilde{j}_{n,m} < \cdots$ と番号付けする．固有値は $\lambda_{n,m} = (\tilde{j}_{n,m}/a)^2$ で与えられ，固有関数は $u_{0,m} = J_0\left(\sqrt{\lambda_{0,m}(x^2+y^2)}\right)$，$u_{n,m} = J_n\left(\sqrt{\lambda_{n,m}(x^2+y^2)}\right)(A_n \cos n\theta + B_n \sin n\theta)$ $(n \geq 1)$ で与えられる．

第 5 章 **5.1** (1) $\displaystyle\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2\ell}$, $\displaystyle b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{2\ell}\,dx$, $n = 1,\ 2,\ \dots$.

(2) $\displaystyle\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2\ell}$, $\displaystyle a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{2\ell}\,dx$, $n = 0,\ 1,\ 2,\ \dots$.

5.2 (1) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}2\ell^2}{n\pi} + \frac{4\{(-1)^n - 1\}\ell^2}{n^3\pi^3}\right]\sin\frac{n\pi x}{\ell}$.

(2) $\displaystyle\sum_{n=1}^{\infty}\left[\left(\frac{2}{n\pi} - \frac{4\ell^2}{n^3\pi^3}\right)\{1 - (-1)^n\} + \frac{2(-1)^{n+1}\ell(\ell+2)}{n\pi}\right]\sin\frac{n\pi x}{\ell}$.

(3) $\displaystyle\sum_{n=2}^{\infty}\frac{2\{1 + (-1)^n\}n}{(n^2-1)\pi}\sin\frac{n\pi x}{\ell} = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{(4m^2-1)}\sin\frac{2mx}{\ell}$.

(4) $\displaystyle\sum_{n=1}^{\infty}\frac{2n\pi\{1 - (-1)^n e^{\ell}\}}{n^2\pi^2 + \ell^2}\sin\frac{n\pi x}{\ell}$.

(5) $\displaystyle\frac{\ell}{2}\sin\frac{\pi x}{\ell} - \sum_{n=2}^{\infty}\frac{4n\ell\{1 + (-1)^n\}}{(n^2-1)^2\pi^2}\sin\frac{n\pi x}{\ell} = \frac{\ell}{2}\sin\frac{\pi x}{\ell} - \frac{8\ell}{\pi^2}\sum_{m=1}^{\infty}\frac{m}{(2m-1)(2m+1)}\sin\frac{2m\pi x}{\ell}$.

(6) $\displaystyle\sum_{n=1}^{\infty}\left[\frac{4n\pi\ell^2\{(-1)^n e^{\ell} - 1\}}{(n^2\pi^2 + \ell^2)^2} - \frac{(-1)^n 2n\pi\ell e^{\ell}}{n^2\pi^2 + \ell^2}\right]\sin\frac{n\pi x}{\ell}$.

5.3 (1) $\displaystyle\frac{\ell^2}{6} - \sum_{n=1}^{\infty}\frac{2\{(-1)^n + 1\}\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell} = \frac{\ell^2}{6} - \frac{\ell^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\frac{2m\pi x}{\ell}$.

(2) $\displaystyle\frac{\ell}{4} + \sum_{n=1}^{\infty}\left[\frac{4\ell}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{2\{1 + (-1)^n\}\ell}{n^2\pi^2}\right]\cos\frac{n\pi x}{\ell}$.

5.4 (1) $\displaystyle\frac{\ell^2 + 3\ell + 3}{3} + \sum_{n=1}^{\infty}\frac{4\ell\{(-1)^n(\ell+1) - 1\}}{n^2\pi^2}\cos\frac{n\pi x}{\ell}$.

(2) $\displaystyle \frac{2}{\pi}-\sum_{n=2}^{\infty}\frac{2\{1+(-1)^n\}}{(n^2-1)\pi}\cos\frac{n\pi x}{\ell}=\frac{2}{\pi}-\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{4m^2-1}\cos\frac{2m\pi x}{\ell}.$

(3) $\displaystyle \frac{1-e^{-\ell}}{\ell}+\sum_{n=1}^{\infty}\frac{2\ell\{1-(-1)^n e^{-\ell}\}}{n^2\pi^2+\ell^2}\cos\frac{n\pi x}{\ell}.$

(4) $\displaystyle -\frac{2\ell}{\pi^2}+\frac{\ell}{2}\cos\frac{\pi x}{\ell}-\sum_{n=1}^{\infty}\frac{2\{1+(-1)^n\}(n^2+1)\ell}{(n^2-1)^2\pi^2}\cos\frac{n\pi x}{\ell}.$

(5) $\displaystyle \frac{1-e^{-\ell}(1+\ell)}{\ell}+\sum_{n=1}^{\infty}\left[\frac{2(n^2\pi^2-\ell^2)\{(-1)^n e^{-\ell}-1\}}{(n^2\pi^2+\ell^2)^2}-\frac{2(-1)^n\ell^2 e^{-\ell}}{n^2\pi^2+\ell^2}\right]\cos\frac{n\pi x}{\ell}.$

(6) $\displaystyle \frac{\ell^3}{4}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n 6\ell^3}{n^2\pi^2}+\frac{12\ell^3\{1-(-1)^n\}}{n^4\pi^4}\right]\cos\frac{n\pi x}{\ell}.$

5.5 (1) $\displaystyle \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n\pi}\sin\frac{n\pi x}{\ell}=\frac{1}{2}+\sum_{m=0}^{\infty}\frac{2}{(2m+1)\pi}\sin\frac{(2m+1)\pi x}{\ell}.$

(2) $\displaystyle \frac{\ell}{4}+\sum_{n=1}^{\infty}\left\{\frac{\ell\{(-1)^n-1\}}{n^2\pi^2}\cos\frac{n\pi x}{\ell}-\frac{(-1)^n\ell}{n\pi}\sin\frac{n\pi x}{\ell}\right\}.$

(3) $\displaystyle \frac{\ell}{4}+\sum_{n=1}^{\infty}\left\{\frac{\ell\{1-(-1)^n\}}{n^2\pi^2}\cos\frac{n\pi x}{\ell}+\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}\right\}.$

(4) $\displaystyle \frac{\ell^2}{6}+\sum_{n=1}^{\infty}\left[\frac{(-1)^n 2\ell^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}-\left\{\frac{(-1)^n\ell^2}{n\pi}+\frac{2\{1-(-1)^n\}\ell}{n^2\pi^2}\right\}\sin\frac{n\pi x}{\ell}\right].$

(5) $\displaystyle \frac{1}{\pi}+\frac{1}{2}\sin\frac{\pi x}{\ell}-\sum_{n=2}^{\infty}\frac{1+(-1)^n}{(n^2-1)\pi}\cos\frac{n\pi x}{\ell}.$

(6) $\displaystyle \frac{1}{2}\cos\frac{\pi x}{\ell}+\sum_{n=2}^{\infty}\frac{\{1+(-1)^n\}n}{(n^2-1)\pi}\sin\frac{n\pi x}{\ell}.$

5.6 (1) $\displaystyle \ell-\sum_{n=1}^{\infty}\frac{(-1)^n 2\ell}{n\pi}\sin\frac{n\pi x}{\ell}.$

(2) $\displaystyle \sinh a\ell\left\{\frac{1}{a\ell}+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2+a^2\ell^2}\left(a\ell\cos\frac{n\pi x}{\ell}-n\pi\sin\frac{n\pi x}{\ell}\right)\right\}.$

(3) $\displaystyle \sinh\ell\left\{\frac{1}{\ell}+2\sum_{n=1}^{\infty}\frac{(-1)^n\ell}{n^2\pi^2+\ell^2}\cos\frac{n\pi x}{\ell}\right\}.$

(4) $\displaystyle -2\sinh\ell\sum_{n=1}^{\infty}\frac{(-1)^n n\pi}{n^2\pi^2+\ell^2}\sin\frac{n\pi x}{\ell}.$

(5) $\displaystyle 2\ell^3\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n\pi}-\frac{6}{n^3\pi^3}\right)\sin\frac{n\pi x}{\ell}.$

(6) $\displaystyle \frac{12\ell^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\sin\frac{n\pi x}{\ell}.$

(7) $\displaystyle \frac{\ell}{\pi}-\frac{\ell}{2\pi}\cos\frac{\pi x}{\ell}-\sum_{n=2}^{\infty}\frac{(-1)^n 2\ell}{(n^2-1)\pi}\cos\frac{n\pi x}{\ell}.$

(8) $-\dfrac{\ell}{2\pi}\sin\dfrac{\pi x}{\ell}+\displaystyle\sum_{n=2}^{\infty}\dfrac{(-1)^n 2n\ell}{(n^2-1)\pi}\sin\dfrac{n\pi x}{\ell}.$

(9) $-\dfrac{\ell}{\pi}+\dfrac{\ell}{2\pi}\cos\dfrac{\pi x}{\ell}+\ell\sin\dfrac{\pi x}{\ell}+\displaystyle\sum_{n=2}^{\infty}\dfrac{(-1)^n 2\ell}{(n^2-1)\pi}\cos\dfrac{n\pi x}{\ell}.$

5.8 (1) $\ell-\displaystyle\sum_{n=1}^{\infty}\dfrac{2\ell}{n\pi}\sin\dfrac{n\pi x}{\ell}.$

(2) $\ell+\displaystyle\sum_{n=1}^{\infty}\dfrac{2\ell}{n\pi}\sin\dfrac{n\pi x}{\ell}.$

(3) $\dfrac{2}{\pi}-\displaystyle\sum_{m=1}^{\infty}\dfrac{4}{(4m^2-1)\pi}\cos\dfrac{m\pi x}{\ell}.$

(4) $\dfrac{2}{\pi}-\displaystyle\sum_{n=1}^{\infty}\dfrac{4(-1)^n}{(4n^2-1)\pi}\cos\dfrac{2n\pi x}{\ell}.$

(5) $\dfrac{e^{2\ell}-1}{2\ell}+\displaystyle\sum_{m=1}^{\infty}\dfrac{\ell(e^{2\ell}-1)}{m^2\pi^2+\ell}\cos\dfrac{m\pi x}{\ell}-\displaystyle\sum_{n=1}^{\infty}\dfrac{n\pi(e^{2\ell}-1)}{n^2\pi^2+\ell}\sin\dfrac{n\pi x}{\ell}.$

(6) $\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{n\pi}\{(-1)^n-1\}\sin\dfrac{n\pi x}{\ell}=-\displaystyle\sum_{m=0}^{\infty}\dfrac{2}{(2m+1)\pi}\sin\dfrac{(2m+1)\pi x}{\ell}.$

(7) $\dfrac{\ell}{2}+\displaystyle\sum_{n=1}^{\infty}\dfrac{2\ell}{n^2\pi^2}\{(-1)^n-1\}\,n^2\pi^2\cos\dfrac{n\pi x}{\ell}=\dfrac{\ell}{2}+\displaystyle\sum_{m=0}^{\infty}\dfrac{4\ell}{(2m+1)^2\pi^2}\cos\dfrac{(2n+1)\pi x}{\ell}.$

5.17 $\dfrac{\ell}{\pi}\left[\dfrac{\pi a_0}{2}+\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n b_n}{n}+\displaystyle\sum_{n=1}^{\infty}\left\{-\dfrac{b_n}{n}\cos\dfrac{n\pi x}{\ell}+\dfrac{a_n+(-1)^{n+1}a_0}{n}\sin\dfrac{n\pi x}{\ell}\right\}\right].$

5.18 $\dfrac{L}{2\pi}\left[\dfrac{\pi a_0}{2}+\displaystyle\sum_{n=1}^{\infty}\dfrac{b_n}{n}+\displaystyle\sum_{n=1}^{\infty}\left\{-\dfrac{b_n}{n}\cos\dfrac{2n\pi x}{L}+\dfrac{a_n-a_0}{n}\sin\dfrac{2n\pi x}{L}\right\}\right].$

第 6 章 **6.1** $u(x,t)=\displaystyle\sum_{n=1}^{\infty}e^{-(n\pi/\ell)^2kt}b_n\sin\dfrac{n\pi x}{\ell},\ \ b_n=\dfrac{2}{\ell}\int_0^{\ell}f(x)\sin\dfrac{n\pi x}{\ell}\,dx.$

6.2 (1) $\dfrac{3}{4}e^{-kt}\sin x-\dfrac{1}{4}e^{-9kt}\sin 3x.$

(2) $\dfrac{8}{\pi}\displaystyle\sum_{m=1}^{\infty}e^{-k(2m+1)^2t}\dfrac{\sin(2m+1)x}{(2m+1)^3}.$

(3) $\dfrac{4}{\pi}\displaystyle\sum_{m=0}^{\infty}e^{-k(2m+1)^2t}\dfrac{(-1)^m\sin(2m+1)x}{(2m+1)^2}.$

6.3 $u(x,t)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}e^{-(n\pi/\ell)^2kt}a_n\cos\dfrac{n\pi x}{\ell},\ \ a_n=\dfrac{2}{\ell}\int_0^{\ell}f(x)\cos\dfrac{n\pi x}{\ell}.$

6.4 (1) $e^{-(\pi/\ell)^2kt}\cos\dfrac{\pi x}{\ell}+1.$

(2) $\dfrac{\ell^3}{3}-\dfrac{2\ell^2}{\pi^3}\displaystyle\sum_{m=0}^{\infty}e^{-k(2m+1)^2t/\ell^2}\dfrac{(-1)^m}{(2m+1)^3}\cos\dfrac{(2m+1)\pi x}{2}$

6.5　f のフーリエ係数を $\{a_n,\, b_n\}$ とすれば

$$u(r,\theta)=\frac{a_0(\log\rho_2-\log r)}{2(\log\rho_2-\log\rho_1)}+\sum_{n=1}^{\infty}\left(\frac{\rho_2}{r}\right)^n\frac{r^{2n}-\rho_1^{2n}}{\rho_2^{2n}-\rho_1^{2n}}(a_n\cos n\theta+b_n\sin n\theta).$$

6.6　g のフーリエ係数を $\{a_n,\, b_n\}$ とすれば

$$u(r,\theta)=\frac{a_0(\log r-\log\rho_1)}{2(\log\rho_2-\log\rho_1)}+\sum_{n=1}^{\infty}\left(\frac{\rho_1}{r}\right)^n\frac{\rho_2^{2n}-r^{2n}}{\rho_2^{2n}-\rho_1^{2n}}(a_n\cos n\theta+b_n\sin n\theta).$$

6.8　$u(r,\theta)=A_0+\rho\displaystyle\sum_{n=1}^{\infty}\frac{\rho^n}{nr^n}(a_n\cos n\theta+b_n\sin n\theta)$．ここに，$A_0$ は任意定数，$a_n,\, b_n$ は $f(\theta)$ のフーリエ係数.

6.9　$u(r,\theta)=\displaystyle\sum_{n=1}^{\infty}b_n\left(\frac{r}{\rho}\right)^n\sin n\theta$，ただし b_n は $f(\theta)$ のフーリエ正弦係数.

6.10　$u(r,\theta)=\dfrac{\alpha\rho}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{b_n}{n}\left(\frac{r}{\rho}\right)^{n\pi/\alpha}\sin\frac{n\pi\theta}{\alpha}$，ただし b_n は $[0,\alpha]$ 上の $f(\theta)$ のフーリエ正弦係数.

6.11　$u(r,\theta)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}a_n\left(\frac{r}{\rho}\right)^{n\pi/\alpha}\cos\frac{n\pi\theta}{\alpha}$，ただし a_n は $[0,\alpha]$ 上の $f(\theta)$ のフーリエ余弦係数.

6.12　$u(x,y)=\displaystyle\sum_{n=1}^{\infty}\frac{b_n}{\sinh\frac{n\pi a}{b}}\,\sinh\frac{n\pi(a-x)}{b}\sin\frac{n\pi y}{b}$，ただし b_n は $[0,\, b]$ 上の $g(y)$ のフーリエ正弦係数.

6.13　$\displaystyle\sum_{m=0}^{\infty}\frac{8}{(4m^2-1)(2m+3)\pi}\cdot\frac{\sin(2m+1)x\ \sinh\{(2m+1)(y-\pi)\}}{\sinh(2m+1)\pi}$

6.14　$\displaystyle\sum_{n=1}^{\infty}b_ne^{-n\pi y/\ell}\sin\frac{n\pi x}{\ell}$，ただし b_n は $f(x)$ の $[0,\,\ell]$ 上のフーリエ正弦係数.

6.15　$u(x,y)=u_1(x,y)+u_2(x,y)$，$u_1(0,y)=g(y)$，$u_2(x,0)=f(x)$，他の境界条件はすべてゼロ，とすれば，

$$u_1=\sum_{n=0}^{\infty}\frac{b_n\cosh\sqrt{\lambda_n}(x-a)\sin\sqrt{\lambda_n}y}{\cosh a\sqrt{\lambda_n}},\quad \lambda_n=\left\{\frac{(n+1/2)\pi}{b}\right\}^2,\ b_n=\frac{2}{b}\int_0^b g(y)\sin\sqrt{\lambda_n}y\,dy,$$
$$u_2=\sum_{n=0}^{\infty}\frac{a_n\sin\sqrt{\mu_n}x\cosh\sqrt{\mu_n}(y-b)}{\cosh b\sqrt{\mu_n}},\quad \mu_n=\left\{\frac{(n+1/2)\pi}{a}\right\}^2,\ a_n=\frac{2}{a}\int_0^a f(x)\sin\sqrt{\mu_n}x\,dx.$$

6.16　$u(x,y)=\dfrac{a_0}{2}+\displaystyle\sum_{n=1}^{\infty}\frac{a_n}{\sinh\frac{n\pi b}{a}}\cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}$，$a_n$ は $f(x)$ の $[0,a]$ 上のフーリエ余弦係数.

6.18　$u(x,t)=\displaystyle\sum_{n=0}^{\infty}\frac{b_n}{c\sqrt{\lambda_n}}\sin c\sqrt{\lambda_n}t\sin\sqrt{\lambda_n}x=\sum_{n=0}^{\infty}\frac{b_n}{2c\sqrt{\lambda_n}}\Big\{\cos\sqrt{\lambda_n}(x-ct)-\cos\sqrt{\lambda_n}(x+ct)\Big\}$，ここに，$\lambda_n=\left\{\dfrac{(n+\frac12)\pi}{\ell}\right\}^2$，$b_n=\dfrac{2}{\ell}\displaystyle\int_0^{\ell}g(x)\sin\sqrt{\lambda_n}x\,dx$．$g(x)=x(\cos\frac{n\pi x}{\ell}+1)$ のとき，次のように $|b_n|/\sqrt{\lambda_n}$ は n とともに急激に減衰する：$b_n=-\dfrac{(-1)^n\ell}{\pi^2}\dfrac{6n^2+6n-\frac12}{\left(n^2-\frac14\right)^2\left(n+\frac32\right)^2}$.

6.21　$h(x,t)$ を $[-\ell,\,\ell]$ 上に奇関数として拡張した関数を $\widetilde{h}(x,t)$ とし，$K(x,t)$ で p.115，命題 6.1 の

熱核を $K(x,t)$ とすれば，$u(x,t)=\displaystyle\int_0^t ds\int_{-\ell}^{\ell} K(x-y,t-s)\widetilde{h}(y,s)\,dy$ と与えられる．

6.22 $g_0(t)$，$g_\ell(t)$ は C^1 級で整合条件 $g_0'(t)=h(0,t)$，$g_\ell'(t)=h(\ell,t)$ を満たすとする．$h_1(x,t)=h(x,t)-\Big[g_0'(t)+\dfrac{x}{\ell}\big\{g_\ell'(t)-g_0'(t)\big\}\Big]$ に対して $[-\ell,\ell]$ 上に奇関数として拡張した関数を $\widetilde{h}_1(x,t)$ とすれば，$u(x,t)=\displaystyle\int_0^t ds\int_{-\ell}^{\ell} K(x-y,t-s)\widetilde{h}_1(y,s)\,dy$ と与えられる．

6.23 $u(x,t)=\displaystyle\sum_{n=1}^{\infty} b_n T_n(t)\sin\frac{n\pi x}{\ell}$，ここに b_n は $f(x)$ の $[0,\ell]$ 上のフーリエ正弦係数，$T_n(t)$ は自然数 n が (i) $n<\dfrac{\pi c}{b\ell}$，(ii) $n=\dfrac{\pi c}{b\ell}$，(iii) $n>\dfrac{\pi c}{b\ell}$，それぞれの場合に応じて次で与えられる：

$$
\begin{aligned}
&\text{(i)}\quad e^{-bt}\left\{\cosh t\sqrt{b^2-(cn\pi/\ell)^2}+\frac{b\sinh t\sqrt{b^2-(cn\pi/\ell)^2}}{\sqrt{b^2-(cn\pi/\ell)^2}}\right\},\\
&\text{(ii)}\quad e^{-bt}(1+bt),\\
&\text{(iii)}\quad e^{-bt}\left\{\cos t\sqrt{(cn\pi/\ell)^2-b^2}+\frac{b\sin t\sqrt{(cn\pi/\ell)^2-b^2}}{\sqrt{(cn\pi/\ell)^2-b^2}}\right\}.
\end{aligned}
$$

6.24 $u(x,t)=\displaystyle\sum_{n=1}^{\infty} b_n T_n(t)\sin\frac{n\pi x}{\ell}$，ここに b_n は $g(x)$ の $[0,\ell]$ フーリエ正弦係数，$T_n(t)$ は 6.23 と同じ n に関する場合分けにより次で与えられる：

$$
\text{(i)}\quad \frac{e^{-bt}\sinh t\sqrt{b^2-(cn\pi/\ell)^2}}{\sqrt{b^2-(cn\pi/\ell)^2}},\qquad \text{(ii)}\quad -\frac{e^{-bt}}{b},\qquad \text{(iii)}\quad \frac{e^{-bt}\sin t\sqrt{(cn\pi/\ell)^2-b^2}}{\sqrt{(cn\pi/\ell)^2-b^2}}.
$$

6.25 $u(x,t)=a_0+a_1e^{-t}(1+t)\cos x+e^{-t}\displaystyle\sum_{n=2}^{\infty} a_n\left(\cos\sqrt{n^2-1}t+\frac{1}{\sqrt{n^2-1}}\sin\sqrt{n^2-1}t\right)\cos nx.$
ただし，a_n は $f(x)$ の $[0,\pi]$ 上のフーリエ余弦係数.

6.26 $u(x,t)=\displaystyle\sum_{m=0}^{\infty}\frac{8\ell^2}{(2m+1)^3\pi^3}\cos\frac{c(2m+1)^2\pi^2 t}{\ell^2}\,\sin\frac{(2m+1)\pi x}{\ell}$

第 7 章 **7.2** (1) $\hat{f}(\xi)=\sqrt{\dfrac{2}{\pi}}\dfrac{1-\cos a\xi}{\xi^2}$ $(\xi\neq 0)$，$\dfrac{a^2}{\sqrt{2\pi}}$ $(\xi=0)$.

(2) $\dfrac{2\sqrt{2\pi}\cos(a\xi)}{4a^2\xi^2-\pi^2}$ $\left(\xi\neq\pm\dfrac{\pi}{2a}\right)$，$\dfrac{a}{\sqrt{2\pi}}$ $\left(\xi=\pm\dfrac{\pi}{2a}\right)$.

(3) $\dfrac{\sqrt{2}(a^2\xi^2-2)}{\sqrt{\pi}\xi^3}\sin(a\xi)+\dfrac{2\sqrt{2}a\cos(a\xi)}{\sqrt{\pi}\xi}$ $(\xi\neq 0)$，$\dfrac{\sqrt{2}a^3}{3\sqrt{\pi}}$ $(\xi=0)$.

(4) $\dfrac{1}{\sqrt{2\pi}}\cdot\dfrac{4}{a\xi^2}\left\{2\cos\dfrac{a\xi}{2}-\big(\cos(a\xi)+1\big)\right\}$ $(\xi\neq 0)$，a $(\xi=0)$.

7.3 (1) $\dfrac{\sin\xi}{\xi}+i\dfrac{\cos\xi-1}{\xi}$.

(2) $\left(\dfrac{\sin\xi}{\xi}+\dfrac{\cos\xi-1}{\xi^2}\right)+i\left(\dfrac{\cos\xi}{\xi}-\dfrac{\sin\xi}{\xi^2}\right)$.

(3) $-\dfrac{\pi\sqrt{2\pi}}{\xi(\xi^2-4\pi^2)}\big\{\sin\xi+i\big(\cos\xi-1\big)\big\}$.

(4) $\dfrac{\sqrt{2}}{\sqrt{\pi}\xi^2}\left\{2\cos\dfrac{\xi}{2}-\big(\cos\xi+1\big)\right\}+\dfrac{2i}{\xi^2}\left(\sin\xi-2\sin\dfrac{\xi}{2}\right)$.

7.4 (1) $\dfrac{a|\xi|+1}{4a^3}e^{-a|\xi|}$.

(2) $-\dfrac{i\xi}{2\sqrt{2}a}e^{-a|\xi|}$.

(3) $\dfrac{\sqrt{\pi}}{2a^3}e^{-a|\xi|/\sqrt{2}}\left(\cos\dfrac{a|\xi|}{\sqrt{2}}+\sin\dfrac{a|\xi|}{\sqrt{2}}\right)$.

(4) $-i\dfrac{\sqrt{\pi}}{\sqrt{2}a^2}e^{-a|\xi|/\sqrt{2}}\sin\dfrac{a\xi}{\sqrt{2}}$.

(5) $\sqrt{\dfrac{\pi}{2}}\dfrac{1}{\cosh\frac{\pi\xi}{2}}$.

7.5 (1) $\sqrt{\dfrac{\pi}{6}}e^{-\sqrt{3}|\xi|/2}\left(\cos\dfrac{\xi}{2}+i\sin\dfrac{\xi}{2}\right)$.

(2) $\dfrac{\sqrt{\pi}}{2\sqrt{2}}\left(e^{-|\xi-1|}+e^{-|\xi+1|}\right)$.

(3) $\dfrac{\sqrt{\pi}}{2\sqrt{2}i}\left(e^{-|\xi-1|}-e^{-|\xi+1|}\right)$.

7.6 $\dfrac{i^n n!}{a^{n+1}}\sqrt{\dfrac{2}{\pi}}\cdot\cos^{n+1}\left(\arctan\dfrac{\xi}{a}\right)\cdot\sin\left(\arctan\dfrac{\xi}{a}+\dfrac{(n+1)\pi}{2}\right)$.

7.7 $\dfrac{\pi}{2a^3}$.

7.8 (1) $f(x)=\begin{cases}2-|x| & |x|\le 2,\\ 0, & |x|>2.\end{cases}$

(2) $\dfrac{2\pi a}{x^2+4a^2}$.

(3) $f(x)=\begin{cases}xe^{-x}, & x\ge 0,\\ 0, & x<0.\end{cases}$

(4) $\sqrt{\dfrac{\pi}{2}}\exp\left(-\dfrac{x^2}{2}\right)$.

(5) $f(x)=\begin{cases}0, & x>2b,\ \text{または},\ x<2a,\\ x-2a, & 2a\le x<a+b,\\ 2b-x, & a+b\le x\le 2b.\end{cases}$

7.9 (1) $\widehat{f}(\xi)=\begin{cases}\dfrac{2\sqrt{2}}{\sqrt{\pi}}\dfrac{\sin^2\xi}{\xi^2}, & \xi\ne 0,\\ \dfrac{2\sqrt{2}}{\sqrt{\pi}}, & \xi=0.\end{cases}$ これから $I=\dfrac{2\pi}{3}$.

(2) $\widehat{f}(\xi)=\dfrac{\pi\sqrt{\pi}}{\sqrt{2}}e^{-2a|\xi|}$.

(3) $\widehat{f}(\xi)=\dfrac{(1-\xi^2)-2i\xi}{\sqrt{2\pi}(\xi^2+1)^2}$.

(4) $\widehat{f}(\xi)=\sqrt{\dfrac{\pi}{2}}\exp\left(-\dfrac{\xi^2}{2}\right)$.

(5) $\widehat{f}(\xi) = -\dfrac{1}{\sqrt{2\pi}} \dfrac{e^{-2ib\xi} + e^{-2ia\xi} - 2e^{-i(a+b)\xi}}{\xi^2}$ $(\xi \neq 0)$，$\widehat{f}(0) = \dfrac{(b-a)^2}{\sqrt{2\pi}}$.

7.10 $I = \pi a$.

7.11 $f * g(x) = \begin{cases} 0, & x \leq -a, \\ 1 - e^{-(x+a)}, & -a < x \leq a, \\ 2e^{-x} \sinh a, & x > a. \end{cases}$ $\mathscr{F}[f * g](\xi) = \sqrt{\dfrac{2}{\pi}} \dfrac{\sin a\xi}{\xi(1+i\xi)}$.

7.12 (1) $\mathscr{F}_{\mathrm{s}}\big[e^{-ax}\big](\xi) = \sqrt{\dfrac{2}{\pi}} \dfrac{\xi}{\xi^2 + a^2}$，$\mathscr{F}_{\mathrm{c}}\big[e^{-ax}\big](\xi) = \sqrt{\dfrac{2}{\pi}} \dfrac{a}{\xi^2 + a^2}$.

(2) $\mathscr{F}_{\mathrm{s}}\big[xe^{-ax}\big](\xi) = \sqrt{\dfrac{2}{\pi}} \dfrac{2a\xi}{(\xi^2 + a^2)^2}$，$\mathscr{F}_{\mathrm{c}}\big[xe^{-ax}\big](\xi) = \sqrt{\dfrac{2}{\pi}} \dfrac{a^2 - \xi^2}{(\xi^2 + a^2)^2}$.

7.14 (1) $e^{-t} K(\cdot, t) * f(x)$.

(2) $e^{-v^2 t} \displaystyle\int_{-\infty}^{\infty} e^{v(x-y)} K(x-y, t) f(y)\, dy$.

7.15 (1) $e^{(a^2 - b^2)kt + ax} \cos b(2akt + x)$.

(2) $e^{(a^2 - b^2)kt + ax} \sin b(2akt + x)$.

7.16 (1) $e^{-a^2 kt} \cos(ax)$.

(2) $\dfrac{1}{\sqrt{4kt+1}} \exp\left(-\dfrac{x^2}{4kt+1}\right)$.

(3) 相補誤差関数 $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ を複素変数に拡張して用いれば,

$$\frac{\sqrt{\pi}}{2\sqrt{4kt}} \left\{ \exp\left\{-\frac{(x-i)^2}{4kt}\right\} \mathrm{erfc}\left(\frac{1+ix}{\sqrt{4kt}}\right) + \exp\left\{-\frac{(x+i)^2}{4kt}\right\} \mathrm{erfc}\left(\frac{1-ix}{\sqrt{4kt}}\right) \right\}.$$

(4) $\dfrac{1}{2}\left\{1 + \mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right)\right\}$.

7.17 $4ktK(x,t)(x^3 + 10ktx) + (12k^2t^2 + 12ktx^2 + x^4)\, \mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right)$.

7.19 $4kt(x^2 + 4kt)K(x,t) + (x^3 + 6kt)\, \mathrm{erf}\left(\dfrac{x}{\sqrt{4kt}}\right)$.

7.20 $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{ix\xi} \dfrac{\sinh(1-y)\xi}{\sinh\xi} \hat{f}(\xi)\, d\xi + \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{ix\xi} \dfrac{\sinh y\xi}{\sinh\xi} \hat{g}(\xi)\, d\xi$.

7.21 $u = u_1 + u_2$，$u_1(x,y) = \dfrac{y}{\pi} \displaystyle\int_0^{\infty} \left\{ \dfrac{1}{(x-z)^2 + y^2} - \dfrac{1}{(x+z)^2 + y^2} \right\} f(z)\, dz$,

$u_2(x,y) = \dfrac{x}{\pi} \displaystyle\int_0^{\infty} \left\{ \dfrac{1}{x^2 + (y-z)^2} - \dfrac{1}{x^2 + (y+z)^2} \right\} g(z)\, dz$.

7.23 (1) $u(x;\varepsilon) = \dfrac{1}{\pi}\left\{\arctan\dfrac{x-a}{\varepsilon} - \arctan\dfrac{x-b}{\varepsilon}\right\}$.

(2) $\displaystyle\lim_{\varepsilon \to +0} u(x;\varepsilon) = \begin{cases} 0, & x < a,\ \text{または},\ x > b, \\ \frac{1}{2}, & x = a, \\ 1, & a < x < b, \\ \frac{1}{2}. & x = b. \end{cases}$

7.24 (2) $\dfrac{d}{dx}\hat{u}(x,\,\eta) = \dfrac{c - ib\eta}{a}\hat{u}(x,\,\eta)$, $\hat{u}(0,\,\eta) = \hat{\varphi}(\eta)$.

(3) $\widehat{u}(x,\,\eta) = e^{cx/a} \cdot \hat{\varphi}(\eta) \exp\left(-i\,\dfrac{bx}{a}\,\eta\right)$.

(4) $u(x, y) = e^{cx/a} \cdot \varphi\left(\dfrac{ay - bx}{a}\right)$.

第 8 章 **8.1** $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^\infty \exp(-x^2)\,dx$ は相補誤差関数を表す.

(1) $\dfrac{1}{s}\left(1 - e^{-as}\right)$.

(2) $\dfrac{b}{(s-a)^2 + b^2}$.

(3) $\dfrac{s-a}{(s-a)^2 + b^2}$.

(4) $\dfrac{n!}{(s-a)^{n+1}}$.

(5) $\dfrac{2s\omega}{(s^2+\omega^2)^2}$.

(6) $\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$.

(7) $\dfrac{\sqrt{\pi}}{2\sqrt{a}}e^{s^2/(4a)}\,\mathrm{erfc}\left(\dfrac{s}{2\sqrt{a}}\right)$.

(8) $\dfrac{1}{s^2+1}\coth\dfrac{\pi s}{2}$.

(9) $\dfrac{1}{(1-e^{-\pi s})(s^2+1)}$.

(10) $\dfrac{1}{s}\arctan\dfrac{a}{s}$.

(11) $\dfrac{1}{s}e^{s^2/(4a^2)}\,\mathrm{erfc}\left(\dfrac{s}{2a}\right)$.

8.2 $\dfrac{1}{s^2} - \dfrac{2\ell e^{-\ell s}}{s(1-e^{-2\ell s})}$.

8.3 $\dfrac{A}{s^2}\tanh\dfrac{as}{2}$.

8.4 (1) $\dfrac{x}{2a}\sin ax$

(2) $a \neq b$ のとき, $\dfrac{1}{a^2-b^2}\left(\dfrac{\sin bx}{b} - \dfrac{\sin ax}{a}\right)$, $a = b$ のとき, $-\dfrac{x}{2a^2}\cos ax + \dfrac{1}{2a^3}\sin ax$.

(3) $\dfrac{1}{2a^3}(\sinh ax - \sin ax)$.

(4) $\dfrac{1}{3a^2}\left\{e^{-ax} - e^{ax/2}\left(\cos\dfrac{\sqrt{3}ax}{2} - \sqrt{3}\sin\dfrac{\sqrt{3}ax}{2}\right)\right\}$.

(5) $\dfrac{e^{-bx} - e^{-ax}}{x}$. (6) $\dfrac{2(e^{bx} - \cos ax)}{x}$.